NONNEGATIVELY CURVED HOMOGENEOUS METRICS OBTAINED BY SCALING FIBERS OF SUBMERSIONS

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Abstract. We consider invariant Riemannian metrics on compact homogeneous spaces G/H where an intermediate subgroup K between G and H exists, so that the homogeneous space G/H is the total space of a Riemannian submersion. We study the question as to whether enlarging the fibers of the submersion by a constant scaling factor retains the nonnegative curvature in the case that the deformation starts at a normal homogeneous metric. We classify triples of groups (H, K, G) where nonnegative curvature is maintained for small deformations, using a criterion proved by Schwachhöfer and Tapp. We obtain a complete classification in case the subgroup H has full rank and an almost complete classification in the case of regular subgroups.

1. Introduction

The study of manifolds with nonnegative or positive sectional curvature is one of the classical fields of Riemannian geometry. Examples of manifolds which admit metrics of strictly positive curvature are scarce, but if one relaxes the curvature condition to include manifolds of nonnegative curvature, the situation is different. For instance, any compact homogeneous space admits a metric of nonnegative sectional curvature.

The class of manifolds supporting Riemannian metrics of nonnegative curvature is larger; nevertheless, only a few methods are known and it is also of interest to explore the family of all Riemannian metrics with nonnegative curvature on a given manifold. Some results in this connection were recently obtained by Schwachhöfer and Tapp [ST] in the setting of compact homogeneous spaces. They investigate certain deformations of a normal homogeneous metric on a compact homogeneous space G/H within the class of G-invariant metrics.

They prove the following structural result. The family of invariant metrics is star-shaped with respect to any normal homogeneous metric if the symmetric matrices corresponding to invariant metrics are parametrized by their inverses. Thus the problem of determining all invariant metrics with nonnegative curvature reduces to determining how long nonnegative curvature is maintained when deforming along a linear path, starting at a normal homogeneous metric.

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Schwachhöfer and Tapp investigate this problem for the simplest nontrivial case, namely they assume there is an intermediate subgroup K between H and G and study metrics obtained through shrinking or enlarging the fibers of the Riemannian submersion $G/H \to G/K$ by a constant scaling factor. As they point out, a metric obtained by shrinking the fibers can be interpreted as a submersion metric obtained via a Cheeger deformation [Ch] and hence shrinking the fibers always preserves nonnegative curvature. On the other hand, if one enlarges the fibers by a constant scaling factor, whether nonnegative curvature is maintained for small deformations depends on the triple (H, K, G). In [ST] they find a criterion on the Lie algebra level, see Theorem 2.1 below. The preservation of nonnegative curvature under scaling up first appears in Grove and Ziller's paper on Milnor spheres [GZ].

This condition holds in particular if (K, H) is a symmetric pair, an observation which yields a new class of examples for nonnegatively curved metrics. To study the Lie-theoretic condition found by Schwachhöfer and Tapp is interesting in its own right and determining which triples (H, K, G) of compact Lie groups satisfy the criterion turns out to be an intriguing problem. In [KK] we classify all such triples in the special case where G is simple of dimension up to 15. In the present article, we use root space decompositions to study the problem for three classes of examples.

The first class consists of triples (H, K, G) where G/K is a symmetric space with $\mathrm{rk}(G/K) = \mathrm{rk}(G)$ and H arises as the intersection of K with a subgroup of maximal rank in G. The second class consists of all triples (H, K, G) for which $\mathrm{rk}(H) = \mathrm{rk}(K) = \mathrm{rk}(G)$. For those two classes of spaces we prove that enlarging the fibers of the submersion $G/H \to G/K$ maintains nonnegative curvature if and only if (K, H) is a symmetric pair. In the last part of this article we consider more generally the criterion for triples (H, K, G) where H, K are regular subgroups of G. Here there are also examples satisfying the criterion such that (K, H) is not a symmetric pair. Some of these examples were known before by work of [ST], but we also present a new class of examples, see Section 5. We obtain an almost complete classification in this case.

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2. Preliminaries

Let $H \subsetneq K \subsetneq G$ be compact Lie groups and let $\mathfrak{h}, \mathfrak{k}, \mathfrak{g}$, be their respective Lie algebras. Let g_0 be a biinvariant inner product on $\mathfrak{g}$. Let $\mathfrak{s}$ be the complement of $\mathfrak{k}$ in $\mathfrak{g}$ and let $\mathfrak{m}$ be the complement of $\mathfrak{h}$ in $\mathfrak{k}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{s}$ is an orthogonal decomposition with respect to g_0. Set $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{s}$. Then $\mathfrak{p}$ can be identified with the tangent space of the homogeneous manifold G/H at the point $1H$. The restriction of g_0 to $\mathfrak{p}$ defines an Ad_H-invariant inner product and thus a G-invariant Riemannian metric on G/H, which we will also denote by g_0.

For any element X in $\mathfrak{p}$, we write $X = X^{\mathfrak{m}} + X^{\mathfrak{s}}$, where $X^{\mathfrak{m}} \in \mathfrak{m}$ denotes the vertical component and $X^{\mathfrak{s}} \in \mathfrak{s}$ denotes the horizontal component of X. We study G-invariant metrics on G/H which are obtained by deforming the Riemannian metric g_0 on G/H such that the length of vectors which are tangent to fibers of the Riemannian submersion

$$G/H \to G/K \tag{2.1}$$

is scaled by a constant and the length of vectors normal to the fibers is unchanged. That is, we consider the one-parameter family of Riemannian metrics g_t on G/H, where $t \in (-\infty, 1)$,

$$g_t(X, Y) := \frac{1}{1-t} \cdot g_0(X^{\mathfrak{m}}, Y^{\mathfrak{m}}) + g_0(X^{\mathfrak{s}}, Y^{\mathfrak{s}}). \tag{2.2}$$

It is well known that the normal homogeneous metric g_0 has nonnegative curvature. As Schwachhöfer and Tapp point out in [ST], a metric g_t where $t < 0$ can be reinterpreted as a submersion metric using Cheeger's construction [Ch] in the following way, see also [Z, Section 2]. Assume G is equipped with the biinvariant metric g_0 and the homogeneous space K/H is equipped with the metric $\lambda^2 \cdot g_0|_{\mathfrak{k}\times\mathfrak{k}}$. Then an isometric action of K on $G \times K/H$ is defined by $k \cdot (g, aH) = (gk^{-1}, kaH)$. The quotient by this action is diffeomorphic to G/H and the submersion metric corresponds to (2.2) with $t = -1/\lambda^2$. Since the metric on the total space of this submersion has nonnegative curvature, it follows from O'Neill's formula that the metric on the base space has nonnegative curvature.

However, metrics g_t with $t > 0$ do not have an such an interpretation and whether they have nonnegative curvature depends on the triple (H, K, G). Schwachhöfer and Tapp prove the following condition on the Lie algebra level.

Theorem 2.1. [ST]

(1) *The metric g_t has nonnegative curvature for small $t > 0$ if and only if there exists some $C > 0$ such that for all X and Y in $\mathfrak{p}$,*

$$\left|[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}\right| \leq C \left|[X, Y]\right|. \tag{2.3}$$

(2) *In particular, if (K, H) is a symmetric pair, then g_t has nonnegative curvature for small $t > 0$, and in fact for all $t \in (-\infty, 1/4]$.*

The first part of (2) follows immediately from the observation that condition (2.3) always holds when $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, in which case the left hand side of the inequality is obviously zero for all X and Y in $\mathfrak{p}$. In particular, (2.3) holds if $\mathfrak{k}$ is abelian.

When H is trivial the submersion (2.1) becomes $K \to G \to G/K$ and g_t is in fact a left-invariant metric on G. In this case, Schwachhöfer [S] proved g_t has nonnegative curvature for small $t > 0$ only if the semisimple part of $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. In particular, when $\mathfrak{g}$ is simple and $\mathfrak{k}$ is nonabelian, one does not get nonnegative curvature by enlarging the fibers.

Here, a pair (K, H) of compact Lie groups $H \subseteq K$ is said to be a *symmetric pair* if there exists an automorphism σ of the Lie algebra $\mathfrak{k}$ with $\sigma^2 = \mathrm{id}_{\mathfrak{k}}$ whose fixed point set is $\mathfrak{h}$. In this case we also say that $(\mathfrak{k}, \mathfrak{h})$ is a *symmetric pair (of Lie algebras)*.

It is shown in [ST] that a number of examples (H, K, G) satisfy the hypothesis of Theorem 2.1. Among them are the following chains $H \subset K \subset G$, where $[\mathfrak{m}, \mathfrak{m}] \not\subseteq \mathfrak{h}$:

- $\mathsf{SU}(3) \subset \mathsf{SU}(4) \cong \mathsf{Spin}(6) \subset \mathsf{Spin}(7)$,
- $\mathsf{G}_2 \subset \mathsf{Spin}(7) \subset \mathsf{Spin}(p+8)$, where $p \in \{0, 1\}$, and
- $\mathsf{SU}(2) \subset \mathsf{SO}(4) \subset \mathsf{G}_2$, where $\mathsf{SU}(2)$ is contained in $\mathsf{SU}(3) \subset \mathsf{G}_2$.

Notation: We will denote the Lie algebra of the compact Lie group of exceptional type G_2 by $\mathrm{Lie}(\mathsf{G}_2)$ in order to avoid confusion with the notation $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_m$ in Theorem 6.5, by which we denote the simple ideals of a Lie algebra $\mathfrak{g}$.

3. A special class of Lie group triples

In what follows, we will review some general facts about the structure of *real* simple Lie algebras. Let T be a maximal torus of G and let $\mathfrak{t}$ be its corresponding Lie subalgebra. Consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ of G restricted to T. As a T-module, the Lie algebra $\mathfrak{g}$ decomposes into the trivial module $\mathfrak{t}$ and its g_0-orthogonal complement $\mathfrak{t}^\perp$, which further decomposes into a sum of pairwise inequivalent two-dimensional irreducible representations $\mathfrak{g}_\alpha^{\mathbf{R}}$ of T such that for each element $H \in \mathfrak{t}$, $\mathrm{Ad}(\exp(H))$ is the action on $\mathfrak{g}_\alpha^{\mathbf{R}}$ by a rotation of the form

$$\begin{pmatrix} \cos(\alpha(H)) & -\sin(\alpha(H)) \\ \sin(\alpha(H)) & \cos(\alpha(H)) \end{pmatrix}$$

with respect to a suitable basis. Via $\mathrm{Ad}(\exp(H)) = \exp(\mathrm{ad}_H)$ we pass to the Lie algebra level; for each $\mathfrak{g}_\alpha^{\mathbf{R}}$, we may choose a g_0-orthonormal basis (X_α, Y_α) such that

$$\mathrm{ad}_H(X_\alpha) = \alpha(H) Y_\alpha, \qquad \mathrm{ad}_H(Y_\alpha) = -\alpha(H) X_\alpha.$$

Note that $\mathfrak{g}_\alpha^{\mathbf{R}} = \mathfrak{g}_{-\alpha}^{\mathbf{R}}$, in fact, replacing the ordered basis (X_α, Y_α) by (Y_α, X_α) reverses the rotation. Apart from this ambiguity, α is uniquely determined by the invariant subspace $\mathfrak{g}_\alpha^{\mathbf{R}}$. We refer to the spaces $\mathfrak{g}_\alpha^{\mathbf{R}}$ as the *real root spaces* of $\mathfrak{g}$. Let R be the set of nonzero elements $\alpha \in \mathfrak{t}^*$ such that there exists a nontrivial T-module $\mathfrak{g}_\alpha^{\mathbf{R}}$ in $\mathfrak{t}^\perp$. Now choose a vector $v \in \mathfrak{t}$ such that $\alpha(v) \neq 0$ for all $\alpha \in R$. Let $R_+ := \{\alpha \in R \mid \alpha(v) > 0\}$. We say R_+ is the set of *positive roots* of $\mathfrak{g}$ and $R = R_+ \cup -R_+$ is the set of *roots*. Let $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$ denote the complexified Lie algebra. For any $\alpha \in \mathfrak{t}^*$ one defines

$$\mathfrak{g}_\alpha^{\mathbf{C}} = \left\{ X \in \mathfrak{g}^{\mathbf{C}} \mid \mathrm{ad}_H(X) = i\alpha(H) X \text{ for all } H \in \mathfrak{t} \right\}.$$

Whenever $\mathfrak{g}_\alpha^{\mathbf{C}} \neq 0$, we say $\mathfrak{g}_\alpha^{\mathbf{C}}$ is a *root space* of $\mathfrak{g}$. For each $\alpha \in R_+$, define the following elements of $\mathfrak{g}^{\mathbf{C}}$:

$$E_\alpha = X_\alpha - iY_\alpha, \qquad E_{-\alpha} = X_\alpha + iY_\alpha,$$

so that we have $\mathrm{ad}_H(E_{\pm\alpha}) = i\alpha(H)E_{\pm\alpha}$. In this way we obtain the *root space decomposition*

$$\mathfrak{g}^{\mathbf{C}} = (\mathfrak{t} \otimes \mathbf{C}) \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha^{\mathbf{C}}.$$

Of course $\mathfrak{g}_\alpha^{\mathbf{C}} \neq 0$ if and only if $\alpha \in R$. In this case $\mathfrak{g}_\alpha^{\mathbf{C}} = \mathbf{C}E_\alpha$.

Remark 3.1. For each α, the basis vectors X_α, Y_α are determined only up to rotation: for any $t \in \mathbf{R}$ we may replace E_α and $E_{-\alpha}$ by $e^{it}E_\alpha$ and $e^{-it}E_{-\alpha}$, which amounts to replacing X_α and Y_α by

$$\cos(t)X_\alpha + \sin(t)Y_\alpha \quad \text{and} \quad -\sin(t)X_\alpha + \cos(t)Y_\alpha. \tag{3.1}$$

It follows from the Jacobi identity that $[\mathfrak{g}_\alpha^{\mathbf{C}}, \mathfrak{g}_\beta^{\mathbf{C}}] \subseteq \mathfrak{g}_{\alpha+\beta}^{\mathbf{C}}$ for all $\alpha, \beta \in \mathfrak{t}^*$. In particular, if $\alpha + \beta$ is not a root, $[\mathfrak{g}_\alpha^{\mathbf{C}}, \mathfrak{g}_\beta^{\mathbf{C}}] = 0$. On the other hand, we have $[\mathfrak{g}_\alpha^{\mathbf{C}}, \mathfrak{g}_\beta^{\mathbf{C}}] = \mathfrak{g}_{\alpha+\beta}^{\mathbf{C}}$ whenever $\alpha + \beta \neq 0$ [H, Thm. 4.3, Ch. III]. We define the number $N_{\alpha,\beta}$ by $[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in R$.

Let Γ be a subset of R. Then Γ is said to be *symmetric* if whenever $\alpha \in \Gamma$, then $-\alpha \in \Gamma$. The set Γ is said to be *closed* if whenever $\alpha, \beta \in \Gamma$ and $\alpha + \beta \in R$, then $\alpha + \beta \in \Gamma$.

We will now consider examples which arise from a special class of compact symmetric spaces G/K, namely those for which $\mathrm{rk}(G/K) = \mathrm{rk}(G)$, i.e. $\mathfrak{s}$ contains a maximal abelian subalgebra of $\mathfrak{g}$. The simply connected irreducible spaces of this type are $\mathsf{SU}(n)/\mathsf{SO}(n)$, $\mathsf{SO}(2n{+}1)/\mathsf{SO}(n{+}1) \times \mathsf{SO}(n)$, $\mathsf{SO}(2n)/\mathsf{SO}(n) \times \mathsf{SO}(n)$, $\mathsf{Sp}(n)/\mathsf{U}(n)$, $\mathsf{E}_6/\mathsf{Sp}(4)$, $\mathsf{E}_7/\mathsf{SU}(8)$, $\mathsf{E}_8/\mathsf{SO}'(16)$, $\mathsf{F}_4/\mathsf{Sp}(3)\,\mathsf{Sp}(1)$, $\mathsf{G}_2/\mathsf{SO}(4)$, cf. [H, Table V, Ch.X]. For the spaces in this class, the Satake diagram of G/K is the same as the Dynkin diagram of G, but with uniform multiplicity one. The corresponding involution σ on $\mathfrak{g}$ induces an involution of $\mathfrak{g}^{\mathbf{C}}$ which acts as minus identity on $\mathfrak{t} \otimes \mathbf{C}$ and sends each root to its negative. Furthermore, every real root space $\mathfrak{g}_\alpha^{\mathbf{R}}$ is σ-invariant, with a one-dimensional fixed point set. For each basis (X_α, Y_α) we have

$$\sigma(X_\alpha) = X_\alpha, \quad \sigma(Y_\alpha) = -Y_\alpha \tag{3.2}$$

after an appropriate rotation as in (3.1). In particular, for the subalgebra $\mathfrak{k}$, $\mathfrak{k} = \mathrm{span}\{X_\alpha \mid \alpha \in R_+\}$ and for its complement, $\mathfrak{s} = \mathfrak{t} \oplus \mathrm{span}\{Y_\alpha \mid \alpha \in R_+\}$.

Assume $\mathfrak{h}$ is spanned by a subset of $\{X_\alpha \,|\, \alpha \in R_+\}$ and (K, H) is not a symmetric pair. Then the g_0-orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{k}$ contains two elements of the form X_λ, X_μ such that the $\mathfrak{m}$-component of their bracket $[X_\lambda, X_\mu]^{\mathfrak{m}}$ is nonzero (this implies $\lambda + \mu \neq 0$). Since $[X_\lambda, X_\mu] \in \mathfrak{g}_{\lambda+\mu} \oplus \mathfrak{g}_{\lambda-\mu} \oplus \mathfrak{g}_{\mu-\lambda} \oplus \mathfrak{g}_{-\lambda-\mu}$, we may assume (possibly after interchanging λ and μ) that at least one of $\lambda \pm \mu$ is a positive root of $\mathfrak{g}$ for which $X_{\lambda\pm\mu} \in \mathfrak{m}$. Let $\nu = \lambda \pm \mu$, where the sign is chosen such that $\nu \in R_+$ and $X_\nu \in \mathfrak{m}$.

The set of roots which appear as nonzero linear combinations of λ and μ with integral coefficients forms an irreducible rank two root system $R(\lambda,\mu)$ which is of type A_2 or B_2. (No root system of type G_2 ever occurs as a proper subset of an irreducible root system.)

We will now show that we may choose two linearly independent elements $\alpha,\beta\in\{\lambda,\mu,\nu\}$ such that $\alpha-\beta\in R_+$ and neither $\alpha+\beta$ nor $2\alpha-\beta$ is a root. In case $R(\lambda,\mu)$ is of type A_2, choose $\alpha:=\lambda$, $\beta:=\mu$ if $\nu=\lambda-\mu$ and choose $\alpha:=\nu$, $\beta:=\mu$ if $\nu=\lambda+\mu$. Assume $R(\lambda,\mu)$ is of type B_2 and $\nu=\lambda+\mu$. Then either λ and μ are orthogonal short roots, in which case we set $\alpha:=\nu$, $\beta:=\mu$, or λ and μ are of different length and enclose an angle of $\frac{3\pi}{4}$, in which case we choose $\alpha:=\nu$ and $\beta\in\{\lambda,\mu\}$ to be the long root.

Now assume $R(\lambda,\mu)$ is of type B_2 and $\nu=\lambda-\mu$. Then either λ and μ are orthogonal short roots, in which case we set $\alpha:=\lambda$, $\beta:=\nu$, or λ and μ are of different length and enclose an angle of $\frac{\pi}{4}$, in which case we choose $\alpha\in\{\lambda,\mu\}$ to be the long root and $\beta:=\nu$.

For any $H\in\mathfrak{t}$ and for any real constant η, define $X=X_\alpha+H$ and $Y=X_\beta+\eta Y_{\alpha-\beta}$. We show that $[X^{\mathfrak{m}},Y^{\mathfrak{m}}]\neq 0$, and $[X,Y]=0$. We recall,

$$[H,X_\beta]=\beta(H)Y_\beta,\quad [H,Y_{\alpha-\beta}]=-(\alpha(H)-\beta(H))X_{\alpha-\beta}.$$

Furthermore, using the fact that $\alpha+\beta$ is not a root of $\mathfrak{g}$, we obtain

$$\begin{aligned}4[X_\alpha,X_\beta]&=[E_\alpha+E_{-\alpha},E_\beta+E_{-\beta}]=N_{\alpha,-\beta}E_{\alpha-\beta}+N_{-\alpha,\beta}E_{\beta-\alpha}\\&=N_{\alpha,-\beta}(X_{\alpha-\beta}-iY_{\alpha-\beta})+N_{-\alpha,\beta}(X_{\alpha-\beta}+iY_{\alpha-\beta}).\end{aligned}$$

It follows that $N_{\alpha,-\beta}=N_{-\alpha,\beta}\in\mathbf{R}\setminus\{0\}$ and we have shown that

$$[X^{\mathfrak{m}},Y^{\mathfrak{m}}]=[X_\alpha,X_\beta]=\frac{1}{2}N_{\alpha,-\beta}X_{\alpha-\beta}.$$

Similarly, we calculate, using the fact that $2\alpha-\beta$ is not a root of $\mathfrak{g}$,

$$\begin{aligned}4[X_\alpha,Y_{\alpha-\beta}]&=i[E_\alpha+E_{-\alpha},E_{\alpha-\beta}-E_{\beta-\alpha}]=-iN_{\alpha,\beta-\alpha}E_\beta+iN_{-\alpha,\alpha-\beta}E_{-\beta}\\&=-N_{\alpha,\beta-\alpha}(iX_\beta+Y_\beta)+N_{-\alpha,\alpha-\beta}(iX_\beta-Y_\beta).\end{aligned}$$

Since $[X_\alpha,Y_{\alpha-\beta}]\in\mathfrak{s}$ it follows that $N_{\alpha,\beta-\alpha}=N_{-\alpha,\alpha-\beta}\in\mathbf{R}\setminus\{0\}$. We have shown

$$[X_\alpha,Y_{\alpha-\beta}]=-\frac{1}{2}N_{\alpha,\beta-\alpha}Y_\beta.$$

Thus we have

$$\begin{aligned}&[X,Y]=[X_\alpha+H,X_\beta+\eta Y_{\alpha-\beta}]=\\&\qquad=\frac{1}{2}N_{\alpha,-\beta}X_{\alpha-\beta}-\frac{1}{2}\eta N_{\alpha,\beta-\alpha}Y_\beta+\beta(H)Y_\beta-\eta(\alpha(H)-\beta(H))X_{\alpha-\beta}.\end{aligned}$$

Since α and β are linearly independent, there exists an element $H\in\mathfrak{t}$ which solves the equation

$$\frac{1}{2}N_{\alpha,-\beta}N_{\alpha,\beta-\alpha}+2\beta(H)(\alpha(H)-\beta(H))=0.$$

Having chosen such an $H \in \mathfrak{t}$, we set $\eta := -2\beta(H)/N_{\alpha,\beta-\alpha}$, then

$$[X, Y] = 0$$

and, on the other hand, $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} = [X_\alpha, X_\beta]^{\mathfrak{m}} \neq 0$. This proves that condition (2.3) does not hold.

Using Borel-de-Siebenthal theory [O, §3], we can describe the triples (H, K, G) covered by the above calculations more explicitly. Let us assume we have chosen bases X_α, Y_α of the real root spaces of $\mathfrak{g}$ as described above, i.e. such that $\mathfrak{k}$ is spanned by $\{X_\alpha | \alpha \in R_+\}$ and $\mathfrak{h}$ is spanned by $\{X_\alpha | \alpha \in S_+\}$ for some subset $S_+ \subset R_+$. Let $\mathfrak{l}$ be the linear subspace of $\mathfrak{g}$ spanned by $\mathfrak{t} \cup \{X_\alpha | \alpha \in S_+\} \cup \{Y_\alpha | \alpha \in S_+\}$. Consider the complexification $\mathfrak{l}^{\mathbf{C}} \subset \mathfrak{g}^{\mathbf{C}}$. Then $\mathfrak{l}^{\mathbf{C}} = (\mathfrak{t} \otimes \mathbf{C}) \oplus \bigoplus_{\alpha \in S} \mathfrak{g}^{\mathbf{C}}_\alpha$, where $S := S_+ \cup -S_+$. By [O, Prop. 15, 1 §3], $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ if and only if the subset S is closed and symmetric. In our case, S is symmetric by definition. Let us show that it is also closed: Let $\alpha, \beta \in S$ such that $\alpha + \beta \in R$. Following [Wa], we set $|\alpha| := \alpha$ if $\alpha \in R_+$ and $|\alpha| := -\alpha$ if $-\alpha \in R_+$. Furthermore, we are using the convention that whenever $\alpha + \beta \notin R$ we set $N_{\alpha,\beta} = 0$, $E_{\alpha+\beta} = 0$ and whenever $\alpha \notin R$, we set $X_\alpha = 0$. We compute

$$\begin{aligned} 4[X_\alpha, X_\beta] &= [E_\alpha + E_{-\alpha}, E_\beta + E_{-\beta}] \\ &= (N_{\alpha,\beta} E_{\alpha+\beta} + N_{-\alpha,-\beta} E_{-\beta-\alpha}) \\ &\quad + (N_{\alpha,-\beta} E_{\alpha-\beta} + N_{-\alpha,\beta} E_{-\alpha+\beta}) \\ &= 2N_{\alpha,\beta} X_{|\alpha+\beta|} + 2N_{\alpha,-\beta} X_{|\alpha-\beta|}. \end{aligned}$$

Since $N_{\alpha,\beta} \neq 0$ and $\mathfrak{h}$ is spanned by $\{X_\alpha | \alpha \in S_+\}$, this shows that $X_{|\alpha+\beta|} \in \mathfrak{h}$ and hence $\alpha + \beta \in S$. Thus $\mathfrak{l}$ is actually a subalgebra of $\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{l} \cap \mathfrak{k}$. Moreover, $\mathrm{rk}(\mathfrak{l}) = \mathrm{rk}(\mathfrak{g})$ and hence the corresponding subgroup L is closed.

Conversely, let $L \subset G$ be a closed subgroup containing a maximal torus T of G. Let $\sigma\colon G \to G$ be an involution as described above and assume we have chosen a root space decomposition and vectors X_α, Y_α, $\alpha \in R_+$ such that (3.2) holds. Let K be the fixed point set of σ and let $H := K \cap L$. Then the triple $H \subset K \subset G$ is such that $\mathfrak{h}$ is spanned by X_α, $\alpha \in S_+$, for some subset $S_+ \subseteq R_+$. We have proved the following.

Theorem 3.2. *Assume (G, K) is a symmetric pair such that $\mathrm{rk}(G/K) = \mathrm{rk}(G)$ and let $\sigma\colon G \to G$ be the corresponding involution. Let L be a σ-invariant subgroup with $\mathrm{rk}(L) = \mathrm{rk}(G)$, and let $H := K \cap L$. Then the triple (H, K, G) satisfies condition (2.3) if and only if (K, H) is a symmetric pair.*

It is interesting to note that while condition (2.3) fails for the triples $H \subsetneq K \subsetneq G$ where (K, H) is not a symmetric pair, it always holds for the triples $H \subsetneq L \subsetneq G$, since (L, H) is a symmetric pair. Indeed, the involution σ leaves L invariant and thus induces an involution of the same kind as σ (mapping each root to its negative and acting as minus identity on $\mathfrak{t}$) on L.

We do not give a full list of the triples where Theorem 3.2 applies, but just illustrate the result by the following examples.

Corollary 3.3. *The following chains $H \subsetneq K \subsetneq G$ of compact Lie groups do not satisfy condition (2.3) in Theorem 2.1 (1):*

(1) $\mathsf{SO}(n_1)\times\mathsf{SO}(n_2)\times\mathsf{SO}(n_3) \ \subset\ \mathsf{SO}(n) \ \subset\ \mathsf{SU}(n)$, $n_i \geq 1$, $n_1+n_2+n_3 = n$.
(2) $[\mathsf{SO}(n_1+1)\times\mathsf{SO}(n_2)\times\mathsf{SO}(n_3)]\times[\mathsf{SO}(n_1)\times\mathsf{SO}(n_2)\times\mathsf{SO}(n_3)]$ $\subset\ \mathsf{SO}(n+1)\times\mathsf{SO}(n) \ \subset\ \mathsf{SO}(2n+1)$, $n_i \geq 1$, $n_1+n_2+n_3 = n$.
(3) $\mathsf{U}(n_1)\times\mathsf{U}(n_2)\times\mathsf{U}(n_3) \ \subset\ \mathsf{U}(n) \ \subset\ \mathsf{Sp}(n)$, $n_i \geq 1$, $n_1+n_2+n_3 = n$
(4) $[\mathsf{SO}(n_1)\times\mathsf{SO}(n_2)\times\mathsf{SO}(n_3)]\times[\mathsf{SO}(n_1)\times\mathsf{SO}(n_2)\times\mathsf{SO}(n_3)]$ $\subset\ \mathsf{SO}(n)\times\mathsf{SO}(n) \ \subset\ \mathsf{SO}(2n)$, *where* $n_i \geq 1$, $n_1+n_2+n_3 = n$.
(5) $\mathsf{SO}(3)\cdot\mathsf{SO}(3)\cdot\mathsf{SO}(3) \ \subset\ \mathsf{Sp}(4) \ \subset\ \mathsf{E}_6$.
(6) $\mathsf{SO}(3)\cdot\mathsf{SO}(6) \ \subset\ \mathsf{SU}(8)/\{\pm 1\} \ \subset\ \mathsf{E}_7$.
(7) $\mathsf{SO}(3)\cdot\mathsf{Sp}(4) \ \subset\ \mathsf{SO}'(16) \ \subset\ \mathsf{E}_8$.
(8) $\mathsf{SO}(3)\cdot\mathsf{SO}(3) \ \subset\ \mathsf{Sp}(3)\cdot\mathsf{Sp}(1) \ \subset\ \mathsf{F}_4$.

Proof. All examples are constructed in the following manner. Choose a subgroup $L \subset G$ of full rank such that (G, L) is not a symmetric pair[1]. Then determine the subgroup $H \subset L$ (unique up to conjugacy) such that (L, H) is a symmetric pair satisfying $\mathrm{rk}(L/H) = \mathrm{rk}(L)$. Here are the subgroups L chosen in the examples above: (1) $\mathsf{S}(\mathsf{U}(n_1)\times\mathsf{U}(n_2)\times\mathsf{U}(n_3))$; (2) $\mathsf{SO}(2n_1+1)\times\mathsf{SO}(2n_2)\times\mathsf{SO}(2n_3)$; (3) $\mathsf{Sp}(n_1)\times\mathsf{Sp}(n_2)\times\mathsf{Sp}(n_3)$; (4) $\mathsf{SO}(2n_1)\times\mathsf{SO}(2n_2)\times\mathsf{SO}(2n_3)$; (5) $\mathsf{SU}(3)\cdot\mathsf{SU}(3)\cdot\mathsf{SU}(3)$; (6) $\mathsf{SU}(3)\cdot\mathsf{SU}(6)$; (7) $\mathsf{SU}(3)\cdot\mathsf{E}_6$; (8) $\mathsf{SU}(3)\cdot\mathsf{SU}(3)$. See [O, Thm. 16, §3] for regular subgroups. □

4. Subgroups of full rank

In this section, we consider the case of closed subgroups $H \subsetneq K \subsetneq G$ of a simple compact Lie group G such that $\mathrm{rk}(H) = \mathrm{rk}(K) = \mathrm{rk}(G)$. We show in Theorem 4.4 that in this case the triple (H, K, G) satisfies condition (2.3) if and only if (K, H) is a symmetric pair. We will show in Theorem 4.4 below that we may restrict ourselves to chains $H \subsetneq K \subsetneq G$ with $\mathrm{rk}(G) \leq 3$. In the following lemma, we prove our result in this special case.

Lemma 4.1. *For the following chains of compact Lie groups $H \subsetneq K \subsetneq G$ there exist elements $X, Y \in \mathfrak{p}$ such that $[X, Y] = 0$ and $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$.*

(1) $T^3 \ \subset\ \mathsf{S}(\mathsf{U}(3)\times\mathsf{U}(1)) \ \subset\ \mathsf{SU}(4)$,
(2) $\mathsf{U}(2)\times\mathsf{SO}(2) \ \subset\ \mathsf{SO}(6) \ \subset\ \mathsf{SO}(7)$,
(3) $\mathsf{U}(2)\times\mathsf{SO}(2) \ \subset\ \mathsf{SO}(5)\times\mathsf{SO}(2) \ \subset\ \mathsf{SO}(7)$,
(4) $\mathsf{U}(1)\times\mathsf{U}(1)\times\mathsf{U}(1) \ \subset\ \mathsf{U}(3) \ \subset\ \mathsf{Sp}(3)$,
(5a) $\mathsf{Sp}(1)\times\mathsf{U}(1)\times\mathsf{Sp}(1) \ \subset\ \mathsf{Sp}(2)\times\mathsf{Sp}(1) \ \subset\ \mathsf{Sp}(3)$,
(5b) $\mathsf{Sp}(1)\times\mathsf{U}(1)\times\mathsf{U}(1) \ \subset\ \mathsf{Sp}(2)\times\mathsf{U}(1) \ \subset\ \mathsf{Sp}(3)$,
(6) $T^2 \ \subset\ \mathsf{SU}(3) \ \subset\ \mathsf{G}_2$.

Here T^n denotes an n-dimensional torus.

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[1]If (G, L) is a symmetric pair, then (L, H) will be a symmetric pair as well, since the involution corresponding to (G, L) commutes with σ.

Proof. Case (4) is a special case of part (3) in Corollary 3.3. For all other cases we exhibit our vectors X, Y using explicit matrix representations.

We identify $\mathfrak{so}(n)$ with the set of skew-symmetric real $n\times n$-matrices, $\mathfrak{u}(n)$ with the set of Hermitian complex $n\times n$-matrices, $\mathfrak{sp}(n)$ with the set of Hermitian quaternionic $n\times n$-matrices. Let $E_{\nu\mu}$ denote the skew-symmetric matrix with the entry $+1$ in position (ν,μ), the entry -1 in position (μ,ν) and zeros elsewhere, while $F_{\nu\mu}$ denotes the symmetric matrix with the entry 1 in positions (μ,ν) and (ν,μ) and zeros elsewhere.

(1) Let $\mathfrak{h} = \mathfrak{t}^3 = \{\operatorname{diag}(it_1, it_2, it_3, -i(t_1+t_2+t_3)) \mid t_1, t_2, t_3 \in \mathbf{R}\}$,
$$\mathfrak{h} \subset \mathfrak{k} = \left\{ \left(\begin{array}{cc} A & \\ & z \end{array} \right) \middle| A \in \mathfrak{u}(3), z = -\operatorname{tr} A \right\}.$$
We take $X = E_{12} + E_{14}$ and $Y = E_{23} + E_{34}$. Note that $E_{12}, E_{23} \in \mathfrak{m}$ while $E_{14}, E_{34} \in \mathfrak{s}$. Then $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = E_{13} \in \mathfrak{m}$ and $[X, Y] = 0$.

(2) Here
$$\mathfrak{u}(2) = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \mid Y = Y^t \right\} \subset \mathfrak{so}(4),$$
and in this way, $\mathfrak{h} = \mathfrak{u}(2)\oplus\mathfrak{so}(2) \subset \mathfrak{so}(4)\oplus\mathfrak{so}(2) \subset \mathfrak{k} = \mathfrak{so}(6)$. Thus $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2$ where $\mathfrak{m}_1 = \mathfrak{so}(4)\ominus\mathfrak{u}(2)$ and $\mathfrak{m}_2 = \mathfrak{so}(6)\ominus(\mathfrak{so}(4)\oplus\mathfrak{so}(2))$. We take $X = E_{15} - E_{17}$, $Y = E_{25} + E_{27}$, where $E_{15}, E_{25} \in \mathfrak{m}$ and $E_{17}, E_{27} \in \mathfrak{s}$. Then $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = -E_{12}$ has a nonzero $\mathfrak{m}$ component, while $[X, Y] = 0$.

(3) We have $\mathfrak{h} = \mathfrak{u}(2) \oplus \mathfrak{so}(2) \subset \mathfrak{so}(4) \oplus \mathfrak{so}(2) \subset \mathfrak{so}(5) \oplus \mathfrak{so}(2)$, and thus $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2$ where $\mathfrak{m}_1 = \mathfrak{so}(5) \ominus \mathfrak{so}(4)$, and $\mathfrak{m}_2 = \mathfrak{so}(4) \ominus \mathfrak{u}(2)$. We take $X = E_{15} + E_{16}$ and $Y = E_{25} - E_{26}$, where $E_{15},\ E_{25} \in \mathfrak{m}_1$, and $E_{16},\ E_{26} \in \mathfrak{s}$. Then $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = -E_{12}$, which has a nonzero $\mathfrak{m}$-component (in $\mathfrak{m}_2$), and $[X, Y] = 0$.

(4) See Corollary 3.3 (3).

(5) In both cases (5a) and (5b), we have the same $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2$ where $\mathfrak{m}_1 = \mathfrak{sp}(2) \ominus (\mathfrak{sp}(1)\oplus\mathfrak{sp}(1))$ and $\mathfrak{m}_2 = \mathfrak{sp}(1) \ominus \mathfrak{u}(1)$. We may take $X^{\mathfrak{m}} = iF_{12}$ and $Y^{\mathfrak{m}} = kF_{12}$ (both in $\mathfrak{m}_2$). We take $X^{\mathfrak{s}} = E_{23} + iF_{13}$ and $Y^{\mathfrak{s}} = jF_{23} - kF_{13}$ (in both cases). Then $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = -2j(F_{11} - F_{22})$, so that $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} = 2jF_{22} \neq 0$, while $[X, Y] = 0$.

(6) See [KK, Subsection 2.4 (1c)].

□

Remark 4.2. Let $H \subsetneq K \subsetneq G$ be a triple of compact Lie groups and suppose $H' \subsetneq K' \subsetneq G'$ is another triple of compact Lie groups with $G' \subseteq G$ such that for the orthogonal complement $\mathfrak{p}'$ of $\mathfrak{h}'$ in $\mathfrak{g}'$ we have $\mathfrak{p}' \subset \mathfrak{p}$, while for the orthogonal complement $\mathfrak{m}'$ of $\mathfrak{h}'$ in $\mathfrak{k}'$ we have $\mathfrak{m}' \subset \mathfrak{m}$. Then it is sufficient to exhibit a pair of vectors $X, Y \in \mathfrak{p}'$ such that $[X, Y] = 0$ but $[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}]^{\mathfrak{m}'} \neq 0$ in order to show that the triple (H, K, G) does not satisfy condition (2.3): Since $X^{\mathfrak{m}} = X^{\mathfrak{m}'}$, $Y^{\mathfrak{m}} = Y^{\mathfrak{m}'}$, and $[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}] \in \mathfrak{g}'$, we know $[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}]^{\mathfrak{m}'} = [X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}$.

Remark 4.3. Conversely, with the notation as in the remark above, if the triple (H, K, G) satisfies condition (2.3) then the triple (H', K', G') also satisfies condition (2.3). In particular, if the group H is enlarged to H', condition (2.3) is preserved.

In the following, we will consider the case of compact Lie groups $H \subsetneq K \subsetneq G$ where H and K are closed subgroups of full rank in G such that (K, H) is not a symmetric pair. We may chose a maximal torus T of H, which is then also maximal torus of K and G, and consider a root space decomposition with respect to T. Using the notation of Section 3, there are subsets $R_H \subsetneq R_K \subsetneq R$, such that

$$\mathfrak{h}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} \oplus \bigoplus_{\alpha \in R_H} \mathfrak{g}_\alpha, \qquad \mathfrak{k}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} \oplus \bigoplus_{\alpha \in R_K} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}, \mathfrak{t}^{\mathbf{C}}, \mathfrak{m}^{\mathbf{C}}, \mathfrak{s}^{\mathbf{C}}$ denote the complexifications of $\mathfrak{h}, \mathfrak{k}, \mathfrak{m}, \mathfrak{s}, \mathfrak{t}$, respectively. Let $R_{\mathfrak{m}} = R_K \setminus R_H$ and $R_{\mathfrak{s}} = R \setminus R_K$. The scalar product on $\mathfrak{t}^*$ is induced from g_0. The root systems R_H and R_K of H and K, respectively, are symmetric, hence the subsets $R_{\mathfrak{m}}$ and $R_{\mathfrak{s}}$ are also symmetric, see [O, Ch. 1, §3.11].

Theorem 4.4. *Let G be a simple compact Lie group and let $H \subsetneq K \subsetneq G$ be closed subgroups. If $\operatorname{rk}(H) = \operatorname{rk}(K) = \operatorname{rk}(G)$ then either (K, H) is a symmetric pair or there exist elements $X, Y \in \mathfrak{p}$ such that $[X, Y] = 0$ and $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$.*

Proof. Since (K, H) is not a symmetric pair, there exist $\lambda, \mu \in R_{\mathfrak{m}}$ such that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \not\subset \mathfrak{h}^{\mathbf{C}}$. Hence, $0 \neq \mathfrak{g}_{\lambda+\mu} \subset \mathfrak{m}^{\mathbf{C}}$. Let $R(\lambda, \mu)$ denote the subset of R consisting of all roots which are nonzero linear combinations of λ and μ with integer coefficients. Because $R(\lambda, \mu)$ contains the six roots $\mathfrak{g}_{\pm\lambda}, \mathfrak{g}_{\pm\mu}, \mathfrak{g}_{\pm(\lambda+\mu)}$, it is an irreducible root system of rank two.

Since $\mathfrak{g}$ is simple, $\mathfrak{k}^{\mathbf{C}}$ acts effectively on $\mathfrak{s}^{\mathbf{C}}$ and there is a root $\nu \in R_{\mathfrak{s}}$ such that $[\mathfrak{g}_\lambda, \mathfrak{g}_\nu] \neq 0$. Let $R(\lambda, \mu, \nu)$ be the set of all roots which are nonzero linear combinations of λ, μ, ν with integer coefficients. Then $R(\lambda, \mu, \nu)$ is a closed, symmetric subsystem of R and hence a root system of rank two or three, which contains $R(\lambda, \mu)$ as a closed, symmetric, proper subsystem.

We know $\lambda + \nu \in R$, thus the root system $R(\lambda, \mu, \nu)$ is irreducible. Hence the inclusion $R(\lambda, \mu) \subset R(\lambda, \mu, \nu)$ is of one of the following types

$$\begin{array}{llllll} (1) & \mathsf{A}_2 \subset \mathsf{A}_3, & (2) & \mathsf{A}_2 \subset \mathsf{B}_3, & (3) & \mathsf{B}_2 \subset \mathsf{B}_3, \\ (4) & \mathsf{A}_2 \subset \mathsf{C}_3, & (5) & \mathsf{B}_2 \subset \mathsf{C}_3, & (6) & \mathsf{A}_2 \subset \mathsf{G}_2. \end{array} \tag{4.1}$$

(Maximal closed symmetric subsets of irreducible root systems are given in [O, Ch.1, §3.11].)

Let $\mathfrak{g}'$ be the semisimple part of the subalgebra of $\mathfrak{g}$ spanned by the real root spaces $\mathfrak{g}_\alpha^{\mathbf{R}}$, $\alpha \in R(\lambda, \mu, \nu)$ and $\mathfrak{t}$. Then $\mathfrak{g}'$ is in fact a simple Lie algebra isomorphic to $\mathfrak{su}(4)$, $\mathfrak{so}(7)$, $\mathfrak{sp}(3)$, or $\operatorname{Lie}(\mathsf{G}_2)$. Let $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{g}'$ and $\mathfrak{k}' := \mathfrak{k} \cap \mathfrak{g}'$. Then $\mathfrak{m}' := \mathfrak{m} \cap \mathfrak{g}'$ is the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{k}$ and $\mathfrak{s}' := \mathfrak{s} \cap \mathfrak{g}'$ is the orthogonal complement of $\mathfrak{k}'$ in $\mathfrak{g}'$. In particular, Remark 4.2 applies.

We complete the proof by showing that for each type of inclusion $R(\lambda,\mu) \subset R(\lambda,\mu,\nu)$ as enumerated in (4.1), the triple $\mathfrak{h}' \subsetneq \mathfrak{k}' \subsetneq \mathfrak{g}'$ corresponds to one of the triples in Lemma 4.1. Thus there exist $X, Y \in \mathfrak{p}' \subseteq \mathfrak{p}$ such that $[X,Y] = 0$ and $[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}]^{\mathfrak{m}'} = [X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$.

Since $\mathfrak{g}_\nu \subset \mathfrak{s}$ we have that $\mathfrak{k}' \subsetneq \mathfrak{g}'$ and since $\mathfrak{g}_{\pm\lambda}, \mathfrak{g}_{\pm\mu}, \mathfrak{g}_{\pm(\lambda+\mu)} \subset \mathfrak{m}$, we know $[\mathfrak{m}', \mathfrak{m}'] \not\subseteq \mathfrak{h}$. In particular $\mathfrak{m}' \neq 0$. Let $\mathfrak{k}^*$ be the subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{t}$ and the real root spaces $\mathfrak{g}_\alpha^{\mathbf{R}}$, $\alpha \in R(\lambda,\mu)$. Since $\lambda, \mu \in R_{\mathfrak{m}}$, it follows that $\mathfrak{k}^*$ is contained in $\mathfrak{k}'$.

When the subalgebra $\mathfrak{k}^* \subsetneq \mathfrak{g}'$ is maximal, it follows that $\mathfrak{k}^* = \mathfrak{k}'$. This is the case for all possible inclusions $\mathfrak{k}^* \subset \mathfrak{g}'$ enumerated in (4.1) except (2) and (5). In case (2) we have $\mathsf{A}_2 \subset \mathsf{A}_3 \subset \mathsf{B}_3$, thus we further distinguish the cases (2a) $\mathfrak{k}' \cong \mathfrak{so}(6)$ and (2b) $\mathfrak{k}' = \mathfrak{k}^* \cong \mathfrak{u}(3)$. In case (5) we distinguish the cases (5a) $\mathfrak{k}' \cong \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ and (5b) $\mathfrak{k}' = \mathfrak{k}^* \cong \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$.

In the cases (1), (4) and (6) where the semisimple part of $\mathfrak{k}'$ is of type A_2, it follows that $\mathfrak{h}'$ is abelian, as all six roots of $R(\lambda,\mu)$ are contained in $R_{\mathfrak{m}}$; hence the triple $\mathfrak{h}' \subset \mathfrak{k}' \subset \mathfrak{g}'$ is determined, up to an automorphism of $\mathfrak{g}'$. We have one of the chains of Lie groups in Lemma 4.1, (1), (4), or (6).

If the semisimple part of $\mathfrak{k}'$ is of type B_2 as in cases (3) and (5), then either six or eight roots of $R(\lambda,\mu)$ are contained in $R_{\mathfrak{m}}$ and it follows that $\mathfrak{h}$ is either abelian or its semisimple part is of type A_1. Note that there are two possibilities for the inclusion $\mathsf{A}_1 \subset \mathsf{B}_2$, corresponding to the inclusions of Lie groups $\mathsf{SU}(2) \subset \mathsf{SO}(5)$ and $\mathsf{SO}(3) \subset \mathsf{SO}(5)$; in the first case the triple $\mathfrak{h}' \subset \mathfrak{k}' \subset \mathfrak{g}'$ corresponds to the chains (3) or (5) in Lemma 4.1. In the second case, the triple $\mathfrak{h}' \subset \mathfrak{k}' \subset \mathfrak{g}'$ would correspond to either $\mathsf{SO}(3) \times \mathsf{SO}(2) \times \mathsf{SO}(2) \subset \mathsf{SO}(5) \times \mathsf{SO}(2) \subset \mathsf{SO}(7)$ in case (3) or $\mathsf{U}(2) \times \mathsf{Sp}(1) \subset \mathsf{Sp}(2) \times \mathsf{Sp}(1) \subset \mathsf{Sp}(3)$ in case (5). But in both cases (K,H) is a symmetric pair, a contradiction.

In case (2a) we have $\mathfrak{g}' \cong \mathfrak{so}(7)$ and $\mathfrak{k}' \cong \mathfrak{so}(6)$. Hence $\mathfrak{h}' \subset \mathfrak{k}'$ is a subalgebra of rank 3 such that $(\mathfrak{k}', \mathfrak{h}')$ is not a symmetric pair. The only possibilities are $\mathfrak{h}' \cong \mathfrak{u}(2) \oplus \mathfrak{u}(1)$ and $\mathfrak{h}' \cong \mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)$. The first case is covered by Lemma 4.1 (2), the second case by Remark 4.2 and Lemma 4.1 (2).

Finally, in case (2b) we have that $\mathfrak{k}' \cong \mathfrak{u}(3)$, it follows that $\mathfrak{h}$ is abelian, since $\mathfrak{m}'$ is at least 6-dimensional. Hence the triple $\mathfrak{h}' \subset \mathfrak{k}' \subset \mathfrak{g}'$ corresponds to a triple $T^3 \subset \mathsf{U}(3) \subset \mathsf{Spin}(7)$. Using Remark 4.2 once more to replace $\mathsf{Spin}(7)$ by $\mathsf{Spin}(6)$, we see that this case is covered by Lemma 4.1 (1) via the isomorphism $\mathsf{SU}(4) \cong \mathsf{Spin}(6)$.

We have now shown that in each case there exist elements $X, Y \in \mathfrak{p}$ such that $[X,Y] = 0$ and $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$. $\square$

Corollary 4.5. *Let G be a compact Lie group and let $H \subsetneq K \subsetneq G$ be closed subgroups. If $\mathrm{rk}(H) = \mathrm{rk}(K) = \mathrm{rk}(G)$ then the triple satisfies condition (2.3) if and only if for each simple factor G_i of G, at least one of the following holds.*

(i) *$(G_i \cap K, G_i \cap H)$ is a symmetric pair.*
(ii) *$\mathfrak{g}_i \subseteq \mathfrak{k}$.*
(iii) *$\mathfrak{g}_i \cap \mathfrak{k} \subseteq \mathfrak{h}$.*

In particular, if there is a simple factor G_i of G for which none of the above conditions holds, then there exist elements $X, Y \in \mathfrak{p}$ such that $[X, Y] = 0$ and $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$.

Proof. Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and where the $\mathfrak{g}_i$ are the simple factors of $\mathfrak{g}$. Since $\mathrm{rk}(H) = \mathrm{rk}(K) = \mathrm{rk}(G)$, we have that $\mathfrak{k} = \mathfrak{z} \oplus (\mathfrak{g}_1 \cap \mathfrak{k}) \oplus \ldots \oplus (\mathfrak{g}_m \cap \mathfrak{k})$ and $\mathfrak{h} = \mathfrak{z} \oplus (\mathfrak{g}_1 \cap \mathfrak{h}) \oplus \ldots \oplus (\mathfrak{g}_m \cap \mathfrak{h})$. From this fact it is obvious that condition (2.3) holds if and only if it holds for each triple $(H \cap G_i, K \cap G_i, G_i)$ where $i = 1, \ldots, m$. Now the first part of the corollary follows from Theorem 4.4.

Assume there is a simple factor G_i of G for which none of (i), (ii), (iii) above holds. Apply Remark 4.2 to the chain $H \cap G_i \subsetneq K \cap G_i \subsetneq G_i$ to see that the second part of the assertion follows. □

5. New examples

Theorem 5.1. *The chain $\mathsf{SU}(n) \subset \mathsf{SO}(2n) \subset \mathsf{SO}(2n+1)$, $n \geq 2$ satisfies condition (2.3).*

Proof. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{so}(2n+1)$ and $\mathfrak{k} = \mathfrak{so}(2n)$. We introduce a complex structure on $\mathbf{R}^{2n}$ by

$$J = \begin{pmatrix} \boxed{\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}} & & & \\ & \boxed{\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}} \end{pmatrix}. \tag{5.1}$$

Then $\mathfrak{u}(n) = \{A \in \mathfrak{so}(2n) \mid JA = AJ\}$, the subset of all matrices in $\mathfrak{so}(2n)$ which commute with J. The orthogonal complement $\mathfrak{m}_0$ of $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$ is then $\mathfrak{m}_0 = \{M \in \mathfrak{so}(2n) \mid JMJ = M\}$, the subset of all matrices in $\mathfrak{so}(2n)$ which anticommute with J. The one-dimensional ideal which is the center of $\mathfrak{u}(n)$ is $\mathbf{R}J =: \mathfrak{z}$, that is, $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{z}$. The orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$ is $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{z}$. For the orthogonal complement $\mathfrak{s}$ of $\mathfrak{k}$ in $\mathfrak{g}$, we have $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{k}$.

Let $X^{\mathfrak{m}}, Y^{\mathfrak{m}} \in \mathfrak{m}$ and $X^{\mathfrak{s}}, Y^{\mathfrak{s}} \in \mathfrak{s}$. Define $X := X^{\mathfrak{m}} + X^{\mathfrak{s}}$, $Y := Y^{\mathfrak{m}} + Y^{\mathfrak{s}}$ in $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{s}$. Let $W \in \mathfrak{so}(2n)$ be a matrix of rank two and let $x \in \mathbf{R}^{2n}$ be a unit vector such that $x \in (\ker W)^{\perp}$. Let $\lambda := |W(x)|$ and let $y := W(x)/\lambda$. Since W is skew symmetric, we have $\mathrm{im}(W) = (\ker W)^{\perp}$ and $W(v) \perp v$ for all $v \in \mathbf{R}^{2n}$. Hence $(\ker W)^{\perp}$ is spanned by the orthonormal vectors x and y. It follows that

$$W = \lambda(yx^t - xy^t).$$

Define $S(x, y) := yx^t - xy^t$ for $x, y \in \mathbf{R}^{2n}$. We have seen that any rank two matrix in $\mathfrak{so}(2n)$ is given by $S(x, y)$ for some $x, y \in \mathbf{R}^{2n}$. In particular, we have $[X^{\mathfrak{s}}, Y^{\mathfrak{s}}] = S(x, y)$ for some $x, y \in \mathbf{R}^{2n}$.

For $t \in \mathbf{R}$ we may define the pair $(\tilde{X}, \tilde{Y}) \in \mathfrak{p} \times \mathfrak{p}$ by

$$\tilde{X} := \cos(t)X + \sin(t)Y, \quad \tilde{Y} := \cos(t)Y - \sin(t)X.$$

Then we have $[\tilde{X}, \tilde{Y}] = [X, Y]$ and $[\tilde{X}^{\mathfrak{m}}, \tilde{Y}^{\mathfrak{m}}]^{\mathfrak{m}} = [X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}$. In particular, by choosing t suitably, we have $\tilde{Y}^{\mathfrak{z}} = \cos(t)Y^{\mathfrak{z}} - \sin(t)X^{\mathfrak{z}} = 0$. Dropping the tildes, we may therefore assume that $Y^{\mathfrak{z}} = 0$ and hence $Y^{\mathfrak{m}} = Y^{\mathfrak{m}_0}$.

Recall that $\mathsf{SO}(2n)/\mathsf{U}(n)$ is a symmetric space of rank $r := \lfloor \frac{n}{2} \rfloor$, see [H]. Indeed, conjugation by the matrix J defines an involutive automorphism of $\mathfrak{k} = \mathfrak{so}(2n)$ whose $(+1)$-eigenspace is $\mathfrak{h} \oplus \mathfrak{z}$ and whose (-1)-eigenspace is $\mathfrak{m}_0$. Let

$$D := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Consider the maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{m}_0$ defined as follows. For n even, define $\mathfrak{a}$ to be the space of all skew symmetric real $2n \times 2n$-matrices of the form

$$\begin{pmatrix} t_1 D & & & \\ & t_2 D & & \\ & & \ddots & \\ & & & t_r D \end{pmatrix} \tag{5.2}$$

where $t_1, \dots, t_r \in \mathbf{R}$. If n is odd, define $\mathfrak{a}$ to be the space of all skew symmetric real $2n \times 2n$-matrices of the form

$$\left(\begin{array}{cccc|cc} t_1 D & & & & & \\ & t_2 D & & & & \\ & & \ddots & & & \\ & & & t_r D & & \\ \hline & & & & 0 & 0 \\ & & & & 0 & 0 \end{array}\right), \tag{5.3}$$

where $t_1, \dots, t_r \in \mathbf{R}$. Since the isometric action of the group H on $\mathfrak{g}$ given by restriction of the adjoint representation of G leaves $\mathfrak{m}_0$, $\mathfrak{z}$ and $\mathfrak{s}$ invariant, we may replace (X, Y) by $(\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y))$ for any $h \in H$ without limitation of generality, since we have

$$|[\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)]| = |\mathrm{Ad}_h([X, Y])| = |[X, Y]|$$

and

$$\begin{aligned} |[\mathrm{Ad}_h(X)^{\mathfrak{m}}, \mathrm{Ad}_h(Y)^{\mathfrak{m}}]^{\mathfrak{m}}| &= |[\mathrm{Ad}_h(X^{\mathfrak{m}}), \mathrm{Ad}_h(Y^{\mathfrak{m}})]^{\mathfrak{m}}| \\ &= |\mathrm{Ad}_h([X^{\mathfrak{m}}, Y^{\mathfrak{m}}])^{\mathfrak{m}}| \\ &= |[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}| \,. \end{aligned}$$

It follows from the theory of symmetric spaces [H] that the subspace $\mathfrak{a}$ intersects all orbits of the H-action on $\mathfrak{m}_0$, in particular, there is an element $h \in H$ such that $\mathrm{Ad}_h(Y^{\mathfrak{m}}) \in \mathfrak{a}$ and we may henceforth assume $Y^{\mathfrak{m}} \in \mathfrak{a}$, i.e.

$Y^{\mathfrak{m}}$ is of the form (5.2) or (5.3). We define a subalgebra of $\mathfrak{k}$ isomorphic to $r \cdot \mathfrak{so}(4)$ as follows. If n is even, define

$$\mathfrak{k}_1 := \left\{ \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix} \in \mathfrak{so}(4r) \;\middle|\; A_1, \dots, A_r \in \mathfrak{so}(4) \right\}.$$

If n is odd, define

$$\mathfrak{k}_1 := \left\{ \left(\begin{array}{cccc|cc} A_1 & & & & & \\ & A_2 & & & & \\ & & \ddots & & & \\ & & & A_r & & \\ \hline & & & & 0 & 0 \\ & & & & 0 & 0 \end{array}\right) \in \mathfrak{so}(4r+2) \;\middle|\; A_1, \dots, A_r \in \mathfrak{so}(4) \right\}.$$

Let $P\colon \mathfrak{k} \to \mathfrak{k}_1$ be the orthogonal projection from $\mathfrak{k}$ onto $\mathfrak{k}_1$. Note that $X^{\mathfrak{z}}, Y^{\mathfrak{m}} \in \mathfrak{k}_1$. Furthermore, the action of $\mathfrak{k}_1$ on $\mathfrak{k}$ leaves $\mathfrak{k}_1$ and its orthogonal complement invariant, hence we have $P([V, Y^{\mathfrak{m}}]) = [P(V), Y^{\mathfrak{m}}]$ for all $V \in \mathfrak{k}$. We have

$$\begin{aligned} |[X,Y]| \geq \left|[X,Y]^{\mathfrak{k}}\right| \geq \left|P([X,Y]^{\mathfrak{k}})\right| &= |P([X^{\mathfrak{m}}, Y^{\mathfrak{m}}] + [X^{\mathfrak{s}}, Y^{\mathfrak{s}}])| = \\ &= |P([X^{\mathfrak{z}}, Y^{\mathfrak{m}}] + [X^{\mathfrak{m}_0}, Y^{\mathfrak{m}}] + [X^{\mathfrak{s}}, Y^{\mathfrak{s}}])| = \\ &= |[X^{\mathfrak{z}}, Y^{\mathfrak{m}}] + [P(X^{\mathfrak{m}_0}), Y^{\mathfrak{m}}] + P([X^{\mathfrak{s}}, Y^{\mathfrak{s}}])| \end{aligned} \tag{5.4}$$

Now let $P_1, \dots, P_r\colon \mathfrak{k} \to \mathfrak{so}(4)$ be the orthogonal projections onto the direct summands of $\mathfrak{k}_1$ isomorphic to $\mathfrak{so}(4)$ such that $P_{\ell+1}$ maps a matrix $A = (a_{ij}) \in \mathfrak{k}$ to its 4×4-submatrix

$$\begin{pmatrix} a_{4\ell+1,\,4\ell+1} & \cdots & a_{4\ell+1,\,4\ell+4} \\ \vdots & & \vdots \\ a_{4\ell+4,\,4\ell+1} & \cdots & a_{4\ell+4,\,4\ell+4} \end{pmatrix}.$$

Let $J_1 = P_1(J) = \ldots = P_r(J)$ be the complex structure on $\mathbf{R}^4$ defined by (5.1) in the case $n = 2$. From (5.4) we have

$$\left|[X,Y]^{\mathfrak{k}}\right|^2 \geq \sum_{\nu=1}^{r} |[cJ_1, P_\nu(Y^m)] + [P_\nu(X^{\mathfrak{m}_0}), P_\nu(Y^{\mathfrak{m}})] + P_\nu([X^{\mathfrak{s}}, Y^{\mathfrak{s}}])|^2 . \tag{5.5}$$

Since $P_\nu([X^{\mathfrak{s}}, Y^{\mathfrak{s}}])$ is a matrix of rank two or zero, we have $P_\nu([X^{\mathfrak{s}}, Y^{\mathfrak{s}}]) = [U^{\mathfrak{s}'}, V^{\mathfrak{s}'}] = S(x,y)$ for two vectors $x = (x_1, x_2, x_3, x_4),\ y = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4$. We claim that there is a constant $C > 0$ such that

$$\left|[X,Y]^{\mathfrak{k}}\right|^2 \geq C^2 \cdot |[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}|^2$$

for all $X, Y \in \mathfrak{p}$. Each of the r summands on the right hand side in (5.5) is of the form

$$\left|[U^{\mathfrak{z}'}, V^{\mathfrak{m}'}] + [U^{\mathfrak{m}_0'}, V^{\mathfrak{m}'}] + [U^{\mathfrak{s}'}, V^{\mathfrak{s}'}]\right|^2 = \left|[U, V]^{\mathfrak{k}'}\right|^2$$

where $U, V \in \mathfrak{p}' = \mathfrak{m}' + \mathfrak{s}'$ is a pair of vectors for the triple

$$(H', K', G') = (\mathsf{SU}(2), \mathsf{SO}(4), \mathsf{SO}(5))$$

where $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ and $\mathfrak{k}' = \mathfrak{h}' \oplus \mathfrak{m}'$ are orthogonal decompositions and where $\mathfrak{z}$ is spanned by J_1. Hence it suffices to verify the claim for the case $n = 2$. We have $U^{\mathfrak{z}'} = cJ_1$ and $V^{\mathfrak{m}'} = \tau D$ for some $\tau \in \mathbf{R}$. Consequently $[U^{\mathfrak{z}'}, V^{\mathfrak{m}'}] = tE$, where

$$E := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

and $t = 2c\tau$. Furthermore, using the fact that $\mathfrak{m}_0'$ is spanned by D and E, it is easy to verify that for an arbitrary matrix $U^{\mathfrak{m}_0'} \in \mathfrak{m}_0'$ we have $[U^{\mathfrak{m}_0'}, V^{\mathfrak{m}'}] = sF$, where

$$F := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $s \in \mathbf{R}$. After choosing the vectors $U^{\mathfrak{z}'}, U^{\mathfrak{m}_0'}, V^{\mathfrak{m}'}$, the real numbers t and s are uniquely determined. Define $W := [U^{\mathfrak{s}'}, V^{\mathfrak{s}'}] =$

$$\begin{pmatrix} 0 & y_1 x_2 - y_2 x_1 & y_1 x_3 - y_3 x_1 & y_1 x_4 - y_4 x_1 \\ y_2 x_1 - y_1 x_2 & 0 & y_2 x_3 - y_3 x_2 & y_2 x_4 - y_4 x_2 \\ y_3 x_1 - y_1 x_3 & y_3 x_2 - y_2 x_3 & 0 & y_3 x_4 - y_4 x_3 \\ y_4 x_1 - y_1 x_4 & y_4 x_2 - y_2 x_4 & y_4 x_3 - y_3 x_4 & 0 \end{pmatrix}$$

We will finish the proof by showing that there is a constant $C > 0$ such that

$$|[U, V]^{\mathfrak{k}'}|^2 \geq C^2 |[U^{\mathfrak{m}'}, V^{\mathfrak{m}'}]^{\mathfrak{m}'}|^2.$$

Using the Euclidean scalar product given by $\langle A, B\rangle = \operatorname{tr}(A^t B)$ and the corresponding norm on $\mathbf{R}^{4\times 4}$, we have

$$\left|[U^{\mathfrak{m}'}, V^{\mathfrak{m}'}]^{\mathfrak{m}'}\right|^2 = |tE + sF|^2 = 4(s^2 + t^2).$$

Assume that $y = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4$ is a fixed unit vector. We choose the orthonormal basis

$$y,\ e_1 := (-y_4, -y_3, y_2, y_1),\ e_2 := (-y_2, y_1, -y_4, y_3),\ e_3 := (y_3, -y_4, -y_1, y_2)$$

of $\mathbf{R}^4$, and write $x = a_0 y + a_1 e_1 + a_2 e_2 + a_3 e_3$ with $a_j \in \mathbf{R}$. The squared length of W is $\operatorname{tr}((yx^t - xy^t)^t (yx^t - xy^t)) = 2\,|x|\,|y| - 2\langle x, y\rangle = 2(a_1^2 + a_2^2 + a_3^2)$.

The length of the orthogonal projection of W on the linear subspace spanned by E is

$$|y_1x_4 - y_4x_1 + y_2x_3 - y_3x_2| = \left|(-y_4, -y_3, y_2, y_1)(x_1, x_2, x_3, x_4)^t\right| = |a_1|.$$

The length of the orthogonal projection of W on the linear subspace spanned by F is

$$|y_1x_2 - y_2x_1 + y_3x_4 - y_4x_3| = \left|(-y_2, y_1, -y_4, y_3)(x_1, x_2, x_3, x_4)^t\right| = |a_2|.$$

We have, since $|sE| = 2s$ and $|tF| = 2t$,

$$\begin{aligned}\left|[U,V]^{\mathfrak{k}'}\right|^2 &= a_1^2 + a_2^2 + 2a_3^2 + (a_1 \pm 2s)^2 + (a_2 \pm 2t)^2\\ &= 2a_1^2 + 2a_2^2 + 2a_3^2 \pm 4a_1 s \pm 4a_2 t + 4s^s + 4t^2\\ &= 2a_3^2 + 2\left(a_1 \pm s\right)^2 + 2\left(a_2 \pm t\right)^2 + 2s^2 + 2t^2\\ &\geq 2(s^2 + t^2) = \frac{1}{2}\left|sE + tF\right|^2 = \frac{1}{2}\left|[U^{\mathfrak{m}'}, V^{\mathfrak{m}'}]^{\mathfrak{m}'}\right|^2.\end{aligned}$$

This finishes the proof. □

6. Regular subgroups

A closed subgroup K of a compact Lie group G is called a *regular subgroup* if $\mathrm{rk}(C_G(K)) = \mathrm{rk}(G) - \mathrm{rk}(K) + \mathrm{rk}(Z(K))$, where $C_G(K)$ is the centralizer of K in G and $Z(K)$ is the center of K. In this case, we also call the Lie algebra $\mathfrak{k}$ of K a *regular subalgebra* of the Lie algebra $\mathfrak{g}$ of G. When $K \subseteq G$ is a regular subgroup, there exists a maximal torus T of G with Lie algebra $\mathfrak{t}$ such that $\mathfrak{k}$ is spanned by a subset of $\mathfrak{t}$ and the real root spaces $\mathfrak{g}_\alpha^{\mathbf{R}}$, $\alpha \in S$, where S is a symmetric and closed subsystem of the root system R of G. Consider a chain of compact Lie groups $H \subsetneq K \subsetneq G$ where K is a regular subgroup of G. Then H is regular subgroup of G if and only if it is a regular subgroup of K.

Lemma 6.1. *Let $H \subsetneq K \subsetneq G$ be compact Lie groups such that G is simple and H, K are regular subgroups. Let T_1 be a maximal torus of $C_K(H)$. Assume that $(K, H \cdot T_1)$ is not a symmetric pair. Then there exist elements $X, Y \in \mathfrak{p}$ such that $[X, Y] = 0$ and $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$.*

Proof. Let T_2 be a maximal torus of $C_G(K)$. Let $H' := H \cdot T_1 \cdot T_2$, let $K' := K \cdot T_2$, let $G' := G$. Then $\mathrm{rk}(H') = \mathrm{rk}(K') = \mathrm{rk}(G')$. Now apply Corollary 4.5 and Remark 4.2 to the chain $H' \subsetneq K' \subsetneq G'$. □

Proposition 6.2. *Let K be a simple compact Lie group and let $H \subset K$ be such that $C_K(H)$ is of positive dimension and $(K, H \cdot T_1)$ is a symmetric pair for a maximal torus $T_1 \subseteq C_K(H)$. Then $K/(H \cdot T_1)$ is a Hermitian symmetric space and T_1 is one-dimensional.*

Proof. This follows from the classification of symmetric spaces [H]. □

Remark 6.3. Let H, K and T_1 be as in Proposition 6.2. Let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{t}_1 \oplus \mathfrak{m}_0$ be an orthogonal decomposition and let $\mathfrak{m} = \mathfrak{t}_1 \oplus \mathfrak{m}_0$. (It follows that $H \subset K$ is a regular subgroup [H].) Let T_0 be a maximal torus of H. Then $T := T_0 \cdot T_1$ is a maximal torus of K. Consider a root space decomposition of $\mathfrak{k}$ and let R_K denote the set of roots. Let R_H denote the subsystem corresponding to the subgroup H. The set R_H consists of all roots in R_K which vanish on $\mathfrak{t}_1$. Hence it follows that the projection of $[X_\beta, Y_\beta]$ on $\mathfrak{t}_1$ is non-zero for all $\beta \in R_{\mathfrak{m}} := R_K \setminus R_H$.

Proposition 6.4. *Let G be a compact Lie group. Let $H \subsetneq K \subsetneq G$ be connected compact Lie groups such that H, K are regular subgroups. Assume that for each simple ideal I of $\mathfrak{g}$ the condition (2.3) holds for the triple of Lie algebras $(I \cap \mathfrak{h}, I \cap \mathfrak{k}, I)$. Then (2.3) also holds for (H, K, G).*

Proof. Let T_2 be a maximal torus of $C_G(K)$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ be a decomposition into ideals of $\mathfrak{g}$ such that $\mathfrak{g}_0$ is abelian and $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ are simple. Define $\mathfrak{k}_i := \mathfrak{g}_i \cap \mathfrak{k}$ and $\mathfrak{h}_i := \mathfrak{g}_i \cap \mathfrak{h}$. Let $\mathfrak{m}_i$ be the orthogonal complement of $\mathfrak{h}_i$ in $\mathfrak{k}_i$ and let $\mathfrak{p}_i$ be the orthogonal complement of $\mathfrak{h}_i$ in $\mathfrak{g}_i$. Now assume that $(\mathfrak{h}_i, \mathfrak{k}_i, \mathfrak{g}_i)$ satisfies (2.3) with a positive constant C_i, i.e. we have

$$|[X^{\mathfrak{m}_i}, Y^{\mathfrak{m}_i}]^{\mathfrak{m}_i}| \leq C_i \, |[X, Y]| \, .$$

for $i = 1, \ldots, n$. Set $C := \max(C_1, \ldots, C_n)$. Now let $X, Y \in \mathfrak{p}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$ and where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Define $\mathfrak{h}' := \mathfrak{h}_0 + \ldots + \mathfrak{h}_n$ and let $\mathfrak{p}' = \mathfrak{p}_0 + \ldots + \mathfrak{p}_n$ be the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}$. Since $\mathfrak{h}' \subseteq \mathfrak{h}$, it follows that $\mathfrak{p}' \supseteq \mathfrak{p}$ and hence $X, Y \in \mathfrak{p}'$. Let $\mathfrak{k}' := \mathfrak{k}_0 + \ldots + \mathfrak{k}_n$ and let $\mathfrak{m}' := \mathfrak{m}_0 + \ldots + \mathfrak{m}_n$. We have $\mathfrak{m} \subseteq \mathfrak{m}' + \mathfrak{t}_2$. Since $T_2 \subseteq C_G(K)$, it follows that $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = [X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}]$. We write $X = X_0 + \ldots + X_n$ and $Y = Y_0 + \ldots + Y_n$, where $X_i, Y_i \in \mathfrak{g}_i$ and compute

$$\begin{aligned} |[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}}| \leq |[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}]^{\mathfrak{m}'}| = |\sum_{i=0}^{n} [X^{\mathfrak{m}_i}, Y^{\mathfrak{m}_i}]^{\mathfrak{m}_i}| \leq \\ \leq |\sum_{i=1}^{n} C_i [X_i, Y_i]| \leq |\sum_{i=1}^{n} C[X_i, Y_i]| = C|[X, Y]|. \end{aligned}$$

□

We will now prove our classification result for chains of regular subgroups.

Theorem 6.5. *Let G be a compact Lie group. Let $H \subsetneq K \subsetneq G$ be connected compact Lie groups such that H, K are regular subgroups of G. If the triple (H, K, G) satisfies condition (2.3) then for each simple ideal $\mathfrak{g}_i$ of $\mathfrak{g}$ one of the following is true.*

(1) $\mathfrak{g}_i \cap \mathfrak{k} = \mathfrak{g}_i$, *i.e. the simple ideal $\mathfrak{g}_i$ is contained in $\mathfrak{k}$.*
(2) $\mathfrak{g}_i \cap \mathfrak{k} \neq \mathfrak{g}_i$ *and $(\mathfrak{g}_i \cap \mathfrak{k}, \mathfrak{g}_i \cap \mathfrak{h})$ is a symmetric pair, possibly such that $\mathfrak{g}_i \cap \mathfrak{k}$ is contained in $\mathfrak{h}$.*
(3) $\mathfrak{g}_i \cong \mathfrak{so}(2n+1)$, $\mathfrak{g}_i \cap \mathfrak{k} \cong \mathfrak{so}(2n)$ *and $\mathfrak{g}_i \cap \mathfrak{h} \cong \mathfrak{su}(n)$.*

(4) $\mathfrak{g}_i \cong \mathfrak{sp}(n)$ *where each but one simple ideal of* $\mathfrak{g}_i \cap \mathfrak{k}$ *is contained in* $\mathfrak{h}$ *and the one simple ideal not contained in* $\mathfrak{h}$ *is isomorphic to* $\mathfrak{sp}(1)$*, standardly embedded.*

(5) $\mathfrak{g}_i \cong \mathrm{Lie}(\mathsf{G}_2)$*,* $\mathfrak{g}_i \cap \mathfrak{k} \cong \mathfrak{so}(4)$ *and* $\mathfrak{g}_i \cap \mathfrak{h} \cong \mathfrak{su}(2)$ *such that* $\mathfrak{g}_i \cap \mathfrak{h}$ *is contained in a subalgebra* $\mathfrak{su}(3) \subset \mathfrak{g}_i$*.*

Proof. Assume (2.3) holds for the triple (H, K, G). Let T_0 be a maximal torus of H. Let T_1 be a maximal torus of $C_K(H)$ and let T_2 be a maximal torus of $C_G(K)$. Let R denote the root system of G with respect to $T := T_0 \cdot T_1 \cdot T_2$. Generalizing the notation from Section 3, let R_H and R_K denote the closed symmetric subsets of R corresponding to the full rank subgroups $H \cdot T_1 \cdot T_2 \subset G$ and $K \cdot T_2 \subset G$, respectively. Let $R_{\mathfrak{m}} = R_K \setminus R_H$ and $R_{\mathfrak{s}} = R \setminus R_K$.

Consider the decomposition $K = T_K \cdot K_1 \cdot \ldots \cdot K_n$ where T_K is a torus and where the K_i are simple. Let $G = Z \cdot G_1 \cdot \ldots \cdot G_m$, where Z is the center of G and where $G_1, \ldots, G_m$ are the simple factors of G. Since K is a regular subgroup, there is a map $f\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $\mathfrak{k}_i \subseteq \mathfrak{g}_{f(i)}$.

Let $\mathscr{R}$ be the set of roots $\beta \in R_{\mathfrak{m}}$ for which there is an $i \in \{1, \ldots, n\}$ such that $\mathfrak{g}^{\mathbf{R}}_{\beta} \subset \mathfrak{k}_i$, $\mathfrak{k}_i \neq \mathfrak{g}_{f(i)}$ and $(K_i, K_i \cap H)$ is not a symmetric pair. If the set $\mathscr{R}$ is empty then one of the first two conditions in the statement of the theorem holds for each $\mathfrak{g}_i$. Thus we may assume the set $\mathscr{R}$ is non-empty.

If $\beta \in \mathscr{R}$ and $\mathfrak{g}^{\mathbf{R}}_{\beta} \subset \mathfrak{k}_i$, then we may apply Lemma 6.1 and Proposition 6.2 to the chain $\mathfrak{h} \cap \mathfrak{k}_i \subsetneq \mathfrak{k}_i \subsetneq \mathfrak{g}_{f(i)}$, showing that the pair $(K_i, H \cap K_i)$ is as described in Proposition 6.2.

Since $G_{f(i)}$ is simple and $\mathfrak{k}_i \neq \mathfrak{g}_{f(i)}$, it follows that for each $\beta \in \mathscr{R}$ there is at least one root $\alpha \in R_{\mathfrak{s}}$ such that $\alpha + \beta \in R$ or $\alpha - \beta \in R$. Replacing β with $-\beta$, if necessary, we may assume that $\alpha + \beta \in R$. Let $\mathscr{P}$ be the set of all pairs (α, β) such that $\beta \in \mathscr{R}$, $\alpha \in R_{\mathfrak{s}}$ and $\alpha + \beta \in R$. For each pair $(\alpha, \beta) \in \mathscr{P}$, consider the set $R(\alpha, \beta)$ of all linear combinations of α and β with integer coefficients. This set $R(\alpha, \beta)$ is a closed symmetric subsystem of R, thus the elements of $R(\alpha, \beta)$ form a root system of rank two. Since $\beta \in R_{\mathfrak{m}}$, $\alpha, \alpha + \beta \in R_{\mathfrak{s}}$, it follows that the root system $R(\alpha, \beta)$ is irreducible, thus of type A_2, B_2, or G_2.

First assume that among all elements of $\mathscr{P}$ there is at least one pair (α, β) such that $R(\alpha, \beta)$ is of type A_2. Then we have $R(\alpha, \beta) = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by the vectors X_α, Y_α, X_β, Y_β, $X_{\alpha+\beta}$, $Y_{\alpha+\beta}$. The algebra $\mathfrak{g}'$ is isomorphic to $\mathfrak{su}(3)$. We may choose a three-dimensional complex representation of $\mathfrak{g}'$ and assume that $X_\alpha = E_{12}, Y_\alpha = iF_{12}, X_\beta = E_{23}, Y_\beta = iF_{23}, X_{\alpha+\beta} = E_{13}, Y_{\alpha+\beta} = iF_{13}$. Define $X^{\mathfrak{m}'} = E_{23}$, $Y^{\mathfrak{m}'} = iF_{23}$, $X^{\mathfrak{s}'} = E_{12} + E_{13}$, $Y^{\mathfrak{s}'} = i(F_{12} - F_{13})$. We see that $[X^{\mathfrak{m}'} + X^{\mathfrak{s}'}, Y^{\mathfrak{m}'} + Y^{\mathfrak{s}'}] = 0$, while $[X^{\mathfrak{m}'}, Y^{\mathfrak{m}'}] = 2i(F_{22} - F_{33})$. By Remark 6.3 this contradicts the assumption that the triple (H, K, G) satisfies (2.3).

Thus we may assume there is no pair $(\alpha, \beta) \in \mathscr{P}$ such that $R(\alpha, \beta)$ is of type A_2. Hence each such $R(\alpha, \beta)$ is of type B_2 or G_2.

Assume $(\alpha, \beta) \in \mathscr{P}$ is such that $R(\alpha, \beta)$ is of type G_2. Since the root system G_2 does not occur as a proper subsystem of any irreducible root system, it follows that the subalgebra $\mathfrak{g}'$ generated by the vectors X_λ, Y_λ, $\lambda \in R(\alpha, \beta)$ is a simple ideal of $\mathfrak{g}$ isomorphic to $\mathrm{Lie}(\mathsf{G}_2)$. It follows from Lemma 6.1 and Proposition 6.2 that the chain $\mathfrak{h} \cap \mathfrak{g}' \subsetneq \mathfrak{k} \cap \mathfrak{g}' \subsetneq \mathfrak{g}'$ is as described in part (5) of the theorem, since otherwise we find a pair of commuting vectors $X, Y \in \mathfrak{g}'$, such that $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} \neq 0$ by the results of [KK, Subsection 2.4].

Finally, we may assume that all root systems $R(\alpha, \beta)$, $(\alpha, \beta) \in \mathscr{P}$, are of type B_2.

First assume that there is at least one pair $(\alpha, \beta) \in \mathscr{P}$ such that $R(\alpha, \beta)$ is of type B_2 and such that β is a short root. Since $R_K \cap R(\alpha, \beta)$ is a closed symmetric subsystem of $R(\alpha, \beta)$ it follows that $\pm\beta \in R_{\mathfrak{m}}$, while $\lambda \in R_{\mathfrak{s}}$ for all $\lambda \in R(\alpha, \beta) \setminus \{\pm\beta\}$. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by all vectors X_λ, Y_λ where $\lambda \in R(\alpha, \beta)$ and let $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$. The algebra $\mathfrak{g}'$ is isomorphic to $\mathfrak{so}(5)$ and $\mathfrak{k}'$ is contained in the regular subalgebra $\mathfrak{so}(3) \oplus \mathfrak{so}(2)$, it is isomorphic to either $\mathfrak{so}(3)$ or $\mathfrak{so}(3) \oplus \mathfrak{so}(2)$. We may choose a representation of $\mathfrak{g}'$ on $\mathbf{R}^5$ such that E_{ij} with $i = 1, 2$, $j = 3, 4, 5$ represent elements of $\mathfrak{s}$ and $E_{23}, E_{45} \in \mathfrak{t}$. Define $X^{\mathfrak{m}} = E_{12}$ and $X^{\mathfrak{s}} = -E_{24}$, $Y^{\mathfrak{m}} = E_{13}$ and $Y^{\mathfrak{s}} = E_{34}$. Then $[X, Y] = [E_{12} - E_{24}, E_{13} + E_{34}] = 0$, yet $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = [E_{12}, E_{13}] = -E_{23}$. It follows from Remark 6.3 that the triple (H, K, G) does not satisfy (2.3).

Now assume that for all root systems $R(\alpha, \beta)$, $(\alpha, \beta) \in \mathscr{P}$ which are of type B_2 each β is a long root. Then there are the following alternatives. Either among these there is at least one pair $(\alpha, \beta) \in \mathscr{P}$ such that there is a long root $\lambda \in R(\alpha, \beta) \cap R_{\mathfrak{s}}$ or all long roots in $R(\alpha, \beta)$ are contained in R_K for each $(\alpha, \beta) \in \mathscr{P}$ such that $R(\alpha, \beta)$ is of type B_2.

Consider the first case. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by all vectors X_λ, Y_λ where $\lambda \in R(\alpha, \beta)$ and let $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$. The algebra $\mathfrak{g}'$ is isomorphic to $\mathfrak{so}(5)$ and $\mathfrak{k}'$ is one of the subalgebras $\mathfrak{su}(2)$ or $\mathfrak{u}(2)$ of $\mathfrak{g}'$. Choose a representation of $\mathfrak{g}'$ on $\mathbf{R}^5$ such that the subalgebra $\mathfrak{u}(2)$ is represented by the linear combinations of the matrices $E_{23}, E_{24} + E_{35}, E_{34} - E_{25}, E_{45}$. Define $X = X^{\mathfrak{m}} + X^{\mathfrak{s}}$, $Y = Y^{\mathfrak{m}} + Y^{\mathfrak{s}}$ where $X^{\mathfrak{m}} = \frac{1}{2}(E_{25} + E_{34})$ and $Y^{\mathfrak{m}} = \frac{1}{2}(E_{23} + E_{45})$. Take $X^{\mathfrak{s}} = E_{14} + \frac{1}{2}(E_{23} - E_{45})$ and $Y^{\mathfrak{s}} = E_{12} + \frac{1}{2}(E_{25} - E_{34})$. Computing, we see that $[X, Y] = 0$ while $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}] = -\frac{1}{2}(E_{24} - E_{35})$. It follows from Remark 6.3 that the triple (H, K, G) does not satisfy (2.3).

Finally we are left with the case where all root systems $R(\alpha, \beta)$, $(\alpha, \beta) \in \mathscr{P}$ are of type B_2, such that each β is a long root and all long roots in $R(\alpha, \beta)$ are contained in R_K for each $(\alpha, \beta) \in \mathscr{P}$. Fix one such root system $R(\alpha_0, \beta_0)$. Let $\mathfrak{g}'$ be the simple ideal of $\mathfrak{g}$ which contains the vectors X_{β_0}, Y_{β_0}. Then $\mathfrak{g}'$ is a regular subalgebra of $\mathfrak{g}$. We have to show that $\mathfrak{g}'$ is as described in (3) or in (4). Since $R(\alpha_0, \beta_0)$ is of type B_2, it follows that the simple factor of G corresponding to $\mathfrak{g}'$ is of type B_n, C_n, or F_4.

Now the statement of the theorem follows from Lemma 6.6. Indeed, in case $\mathfrak{g}'$ is of type B_n, it follows from Lemma 6.6 that the simple ideal $\mathfrak{g}'$ is

as in item (3) of the theorem. In case $\mathfrak{g}'$ is of type F_4, it follows from the Lemma that $\mathfrak{g}'$ is as described in item (2) of the theorem. If $\mathfrak{g}'$ is of type C_n, we use the following counterexample to show that there can only be one simple ideal of $\mathfrak{k} \cap \mathfrak{g}'$ which is not contained in $\mathfrak{h}$. Consider the chain of Lie algebras $\{0\} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \subset \mathfrak{sp}(2)$. It follows from [S, Lemma 2.2] that a pair of vectors X, Y with $[X, Y] = 0$, while $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} = [X^{\mathfrak{m}}, Y^{\mathfrak{m}}] \neq 0$ exists. Indeed, define e.g.

$$X = \begin{pmatrix} j & j+1 \\ j-1 & i-k \end{pmatrix}, \qquad Y = \begin{pmatrix} -i-k & i \\ i & \frac{i}{2} + j - \frac{k}{2} \end{pmatrix}$$

Then $[X, Y] = 0$, while $[X^{\mathfrak{m}}, Y^{\mathfrak{m}}]^{\mathfrak{m}} = [X^{\mathfrak{m}}, Y^{\mathfrak{m}}] \neq 0$. Using Remark 4.2, it follows that $\mathfrak{g}'$ is as described in item (4) of the theorem. □

To solve the remaining cases in the proof of the above theorem, we have to take a closer look at the root systems of type B_n, C_n and F_4.

Lemma 6.6. *Let G be a simple compact Lie group of type B_n, C_n, or F_4. Let $H \subsetneq K \subsetneq G$ be connected subgroups such that $\mathrm{rk}(H) = \mathrm{rk}(K) = \mathrm{rk}(G)$ and such that (K, H) is a Hermitian symmetric pair. Then one of the following is true, where $R_{\mathfrak{s}}$ and $R_{\mathfrak{m}}$ are defined as in Section 3.*

1. *There is pair of non-orthogonal roots $(\alpha, \beta) \in R_{\mathfrak{s}} \times R_{\mathfrak{m}}$ of the same length.*
2. *There is pair of non-orthogonal roots $(\alpha, \beta) \in R_{\mathfrak{s}} \times R_{\mathfrak{m}}$ such that α is a long root and β is a short root.*
3. *The triple of Lie algebras $(\mathfrak{h}, \mathfrak{k}, \mathfrak{g})$ is isomorphic to*
$$(\mathfrak{u}(n), \mathfrak{so}(2n), \mathfrak{so}(2n+1)).$$
4. *The triple $(\mathfrak{h}, \mathfrak{k}, \mathfrak{g})$ is of the following form: $\mathfrak{g} \cong \mathfrak{sp}(n)$ and for each of the simple factors $\mathfrak{k}_1, \ldots, \mathfrak{k}_f$ of $\mathfrak{k}$ we have either $\mathfrak{k}_j \subseteq \mathfrak{h}$ or $\mathfrak{k}_j \cong \mathfrak{sp}(1)$ and $\mathfrak{k}_j \cap \mathfrak{h} \cong \mathfrak{u}(1)$.*
5. *The triple $(\mathfrak{h}, \mathfrak{k}, \mathfrak{g})$ is isomorphic to $(\mathfrak{so}(8), \mathfrak{so}(9), \mathrm{Lie}(\mathsf{F}_4))$.*

Proof. Assume first G is of type B_n, i.e. $\mathfrak{g} \cong \mathfrak{so}(2n+1)$. The Lie algebra $\mathfrak{k}$ is contained in a maximal subalgebra $\mathfrak{k}'$ of full rank of $\mathfrak{g}$. This maximal subalgebra $\mathfrak{k}'$ is conjugate to $\mathfrak{so}(2\ell) \oplus \mathfrak{so}(2n - 2\ell + 1)$, where $\ell = 1, \ldots, n$, [O, Thm. 16, §3]. Using the notation of [BtD, Prop. 6.5, Ch. V], the long roots of G are given by $\pm\vartheta_\mu \pm \vartheta_\nu$ where $1 \leq \mu < \nu \leq n$, while the short roots are $\pm\vartheta_\nu$ where $1 \leq \nu \leq n$. Then we may assume

$$R_{K'} = \{\pm\vartheta_\mu \pm \vartheta_\nu \mid 1 \leq \mu < \nu \leq \ell \vee \ell < \mu < \nu \leq n\} \cup \{\pm\vartheta_\nu \mid \ell < \nu \leq n\}.$$

and we have for the roots $R_{\mathfrak{s}'} \subseteq R_{\mathfrak{s}}$, corresponding to the orthogonal complement $\mathfrak{s}'$ of $\mathfrak{k}'$ in $\mathfrak{g}$,

$$R_{\mathfrak{s}'} = \{\pm\vartheta_\mu \pm \vartheta_\nu \mid 1 \leq \mu \leq \ell < \nu \leq n\} \cup \{\pm\vartheta_\nu \mid 1 \leq \nu \leq \ell\}.$$

We may assume $\ell < n$ since otherwise (3) holds. Since $H \neq K$, there is an element $\beta \in R_{\mathfrak{m}} \subseteq R_{K'}$. If β is a long root, say $\beta = \pm\vartheta_\mu \pm \vartheta_\nu$, then we may choose an element $\alpha = \pm\vartheta_\kappa \pm \vartheta_\lambda \in R_{\mathfrak{s}'} \subseteq R_{\mathfrak{s}}$ where $\kappa = \mu$ or $\kappa = \nu$ and (1)

holds. If β is a short root $\pm\vartheta_\nu$ for some $\ell < \nu \le n$, then choose $\alpha = \vartheta_1 + \vartheta_\nu$ to show that (2) holds.

Now assume G is of type C_n, i.e. $\mathfrak{g} \cong \mathfrak{sp}(n)$, $n \ge 3$. The Lie algebra $\mathfrak{k}$ is again contained in maximal subalgebra $\mathfrak{k}'$ of full rank of $\mathfrak{g}$. The maximal subalgebras $\mathfrak{k}'$ of full rank in $\mathfrak{sp}(n)$ are conjugate to $\mathfrak{u}(n)$, or $\mathfrak{sp}(\ell)\oplus\mathfrak{sp}(n-\ell)$, where $\ell = 1, \dots, \lfloor\frac{n}{2}\rfloor$, see [O, Thm. 16, §3]. It follows by induction that all simple factors of K are of type A_k or C_k, $k < n$. Now, using the notation from [BtD, Prop. 6.6, Ch. V], the short roots of G are $\pm\vartheta_\mu \pm \vartheta_\nu$ where $1 \le \mu < \nu \le n$ and the long roots are $\pm 2\vartheta_\nu$ where $1 \le \nu \le n$. Assume first $\mathfrak{k}$ is contained in a maximal subalgebra of $\mathfrak{g}$ conjugate to $\mathfrak{u}(n)$. Then we may assume

$$R_{K'} = \{\vartheta_\mu - \vartheta_\nu \mid 1 < \nu, \mu \le n,\ \mu \ne \nu\}.$$

and we have for the roots $R_{\mathfrak{s}'} \subseteq R_{\mathfrak{s}}$, corresponding to the orthogonal complement $\mathfrak{s}'$ of $\mathfrak{k}'$ in $\mathfrak{g}$,

$$R_{\mathfrak{s}'} = \{\pm(\vartheta_\mu + \vartheta_\nu) \mid 1 \le \mu < \nu \le n,\} \cup \{\pm 2\vartheta_\nu \mid 1 \le \nu \le n\}.$$

Since $H \ne K$, there is an element $\beta \in R_{\mathfrak{m}} \subseteq R_{K'}$. Obviously, there is some element $\alpha \in R_{\mathfrak{s}'}$ which is not orthogonal to β. Since $R_{K'}$ contains only short roots, either (1) or (2) holds.

Assume now $\mathfrak{k}$ is not conjugate to a subalgebra of $\mathfrak{u}(n)$, hence contained in a maximal subalgebra of $\mathfrak{g}$ conjugate to $\mathfrak{sp}(\ell)\oplus\mathfrak{sp}(n-\ell)$. We may assume

$$R_{K'} = \{\pm\vartheta_\mu \pm \vartheta_\nu \mid 1 \le \mu < \nu \le \ell \vee \ell < \mu < \nu \le n\} \cup \{\pm 2\vartheta_\nu \mid 1 \le \nu \le n\}.$$

and we have for the roots $R_{\mathfrak{s}'} \subseteq R_{\mathfrak{s}}$, corresponding to the orthogonal complement $\mathfrak{s}'$ of $\mathfrak{k}'$ in $\mathfrak{g}$,

$$R_{\mathfrak{s}'} = \{\pm\vartheta_\mu \pm \vartheta_\nu \mid 1 \le \mu \le \ell < \nu \le n\}.$$

Assume that (4) does not hold. Then there is a simple ideal $\mathfrak{k}_j$ of $\mathfrak{k}$ which is either isomorphic to $\mathfrak{u}(q)$ for some $q \in \{1, \dots, n-1\}$ or isomorphic to some $\mathfrak{sp}(q)$ for some $q \in \{2, \dots, n-1\}$ and such that $\mathfrak{k}_j \cap \mathfrak{h} \ne \mathfrak{k}_j$. It follows from [O, §3] that then $R_{\mathfrak{m}}$ contains a short root. But for any root γ the set $R_{\mathfrak{s}'}$ contains a (short) root which is not orthogonal to γ. Thus (1) holds.

Now assume G is of type F_4. The long roots are $\pm\vartheta_\mu \pm \vartheta_\nu$, $1 \le \mu < \nu \le 4$ and the short roots are given by either $\pm\vartheta_\mu$, $1 \le \mu \le 4$ or $\frac{1}{2}(\pm\vartheta_1 \pm \vartheta_2 \pm \vartheta_3 \pm \vartheta_4)$, see [B] or [O]. The maximal subgroups of maximal rank in F_4 are $\mathsf{Spin}(9)$, $\mathsf{Sp}(3)\cdot\mathsf{Sp}(1)$, and $\mathsf{SU}(3)\cdot\mathsf{SU}(3)$, see [O, Thm. 16, §3].

Assume first that K is conjugate to a subgroup of $\mathsf{SU}(3)\cdot\mathsf{SU}(3)$. We deduce from [O, Thm. 16, §3] that the root system corresponding to this subgroup consists of six short roots and six long roots. The 24 long roots from R comprise a subsystem of type D_4. Thus for any element γ in R there is a long root in $R_{\mathfrak{s}'}$ which is non-orthogonal to γ. Hence (2) holds.

Now assume K is conjugate to a subalgebra of $\mathsf{Sp}(3)\cdot\mathsf{Sp}(1)$. The root system of this group consists of 12 short and 8 long roots. These 8 long roots correspond to a subalgebra of type $4\cdot\mathfrak{sp}(1) \cong \mathfrak{so}(4)\oplus\mathfrak{so}(4) \subset \mathfrak{so}(8)$. A similar argument as above shows that (2) holds.

Finally, assume K is conjugate to a subgroup of $\mathsf{Spin}(9)$. We have

$$R_{K'} = \{\pm\vartheta_\mu \pm \vartheta_\nu \mid 1 \leq \mu < \nu \leq 4\} \cup \{\pm\vartheta_\mu \mid 1 \leq \mu \leq 4\}.$$

If K is any maximal connected subgroup of maximal rank in $\mathsf{Spin}(9)$ then (2) holds, cf. [O, Thm. 16, §3]. Thus we may assume $K = \mathsf{Spin}(9)$. Note that for every root γ in R there is a (short) root in the set $R_{\mathfrak{s}'} = \{\frac{1}{2}(\pm\vartheta_1 \pm \vartheta_2 \pm \vartheta_3 \pm \vartheta_4)\}$ which is not orthogonal to γ. Thus if (1) does not hold then $R_{\mathfrak{m}}$ consists exclusively of long roots. It follows that $H = \mathsf{Spin}(8)$. □

Remark 6.7. For items (1), (2) (3) and (5) in Theorem 6.5 we know that condition (2.3) holds for the chains $(\mathfrak{h} \cap \mathfrak{g}_i, \mathfrak{k} \cap \mathfrak{g}_i, \mathfrak{g}_i)$. However, we conjecture that condition (2.3) holds also for each chain of regular subgroups $(H, K, G) = (\mathsf{Sp}(1)^{n-1}, \mathsf{Sp}(1)^n, \mathsf{Sp}(n))$ with $n \geq 2$. If the conjecture is true, it follows from Proposition 6.4 that the statement in Theorem 6.5 can be improved to "if and only if". To our knowledge, there are no known examples of chains (H, K, G) satisfying condition (2.3) which contain non-regular subgroups, cf. [KK].

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